Embeddings of maximal tori in classical groups, odd descent and Totaro's question.

§ Introduction

Let K be a field of char $\neq 2$.

Let (E, σ) be an étale alg of finit rank n over K with involution σ.
i.e. $\sigma^2 = id$, $\sigma(xy) = \sigma(y)\sigma(x)$

Let (A, τ) be a central simple algebra over K of dim n^2 and involution τ.

Suppose the following conditions hold:

① $\sigma|_K = \tau|_K$

Let $k = K^\sigma$.

② $\dim_k(E^\sigma) = \begin{cases} n & \text{if } [K:k]=2 \\ \lfloor \frac{n+1}{2} \rfloor & \text{if } K=k. \end{cases}$

Q1: $\exists\, \iota : (E,\sigma) \hookrightarrow (A,\tau)$ an inj K-alg homomorphism with respect to involution?

With conditions ① ②, Q1 is "equivalent" to the embedding of maximal tori in classical group.

When K is a global field, one may ask

Q2: Does the local-global principle hold for the embedding problem defined in Q1?

In general, no! (G. Prasad & A. Rapinchuk)

There's some "obstruction" to the local-global principle.

Reformulate Q1 in the language of homogeneous spaces E under

$Aut(G)$, where $G = U(A,\tau)^{0}$

$U(A,\sigma)(R) := \{ x \in A \otimes_k R \mid x\tau(x) = 1 \}$

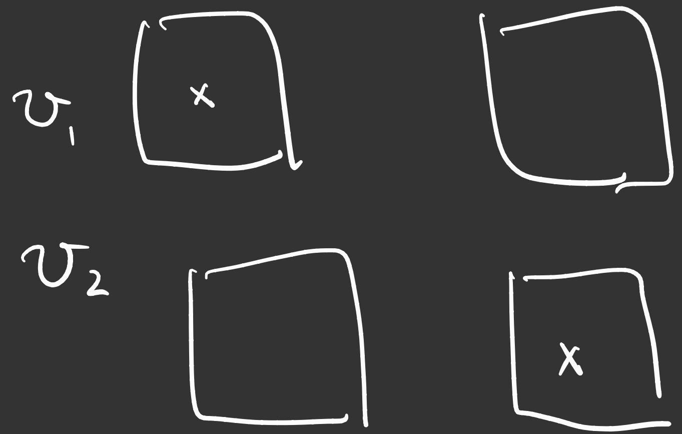
where R is a k-alg.

① an embedding $/k$ $\leftrightsquigarrow$ a k-point of E.

② the geometric stabilizer of $x \in E(k^s)$ is a maximal torus of $Aut(G)_{k^s}$.

Obstruction I:

$Aut(G)$ is disconnected for G of type A_n, D_n.



We must $\boxed{\text{fix}}$ a connected component of E and consider the local-global principle for a connected component.

§ Orientation (for D_n type)

Rmk: For A_n type, a K-algbra embedding lies in the same connected component.

For D_n type (i.e. orthogonal type)

Let $C(A, \tau)$ be the clifford

algebra of (A,τ) and denote by $Z(A,\tau)$ the center of $C(A,\tau)$, which is a quadritic étale alg of k.

An orientation is a k-alg isomophism between $\Delta(E) \xrightarrow{\sim} Z(A,\tau)$.

Fact: ① $(E,\sigma)_v \hookrightarrow (A,\tau)_v$ locally everywhere $\Rightarrow$ $E \hookrightarrow A / k$

② $E \hookrightarrow A \Rightarrow \exists$ an involution θ of A such that

$\epsilon : (E,\sigma) \longrightarrow (A,\theta) \ / k$

③ $(E,\sigma) \hookrightarrow (A,\tau)$

$\Leftrightarrow \exists\, a \in (E^\times)^\sigma$ s.t.

$(A,\tau) \simeq (A,\theta_a)$

$\theta_a(x) := \theta(\epsilon(a)\, x\, \epsilon(a)^{-1})$

Let $\nu : \Delta(E) \to Z(A,\tau)$ be an orientation, and $u : \Delta(E) \longrightarrow Z(A,\theta)$

Thm: (Bayer-L.-Parimala) The following are equivalent: (for D_{2n} type)

(i) There exists an oriented embedding

$$(E,\sigma) \otimes_k k_v \to (A,\tau) \otimes_k k_v$$

with resp. to ν for all $v \in V_k$

(ii) View $C(A \otimes_k k_v, \theta)$ and $C(A \otimes_k k_v, \tau)$ as $\Delta(E)$-module via u, ν respectively.

Then $[C(A \otimes_k k_v, \theta)] = [C(A \otimes_k k_v, \tau)]$ in $Br(\Delta(E \otimes_k k_v))$ for all v in P,

where $P := \{ v \in V_k \mid (A,\tau) \otimes_k k_v \simeq (M_{2n}(H_v), \tau_h)$ and $E \otimes_k k_v$ splits$\}$.

Obstruction II: Brauer-Manin Obstruction

§ Odd descent.

Thm (Bayer-L.-Parimala)

There exists an embedding of (E,σ) into (A,τ) iff such an embedding exists over a finite extension L/K of odd degree.

Key point: (i) $u: \boxed{\Delta E} \otimes_K L \xrightarrow{\sim} Z(A,\tau) \otimes_K L$

$$\Leftrightarrow u : \Delta E \longrightarrow Z(A,\tau)$$

(ii) Odd degree descent holds over local field.

(iii) The Brauer-Manin obstruction vanishes over L iff it vanishes over K.

§ Totaro's question (2004)

If a homogeneous space has a zero cycle of degree one, does it also have a rational point?

Odd degree descent $\Rightarrow$ affirmative answers to Totaro's question.

Prop. (Bayer-L.-Parimala)

Let X be a homogeneous space over a number field under a connected linear algebraic group G with connected stabilizers. Suppose that X has a zero cycle of degree one, then it has a rational point.

Key points

(i) flasque resolution of G:

(Colliot-Thélène)

$1 \to S \to H \to G \to 1$. exact sequence of algebraic groups with the semi-simple part H^{ss} of H simply connected and H^{tor} ($= H^{red}/H^{ss}$) quasi-trivial,

i.e. $H^{tor} \cong \prod_{i=1}^{n} R_{K_i/K}(\mathbb{G}_m)$.

(ii) Brauer-Manin obstruction for X (Borovoi)

For $x \in X(\bar{K})$, $\bar{M} = \mathrm{Stab}_{G_{\bar{K}}}(x)$, there's a K-form M^{tor} of $\bar{M}^{tor}$ and an element $\eta(x) \in H^1(K, M^{tor} \to G^{tor})$ such that if X has a K-point, then

$\eta(x) = 0$

$$\Big(1 \to \underset{-1}{M^{tor}} \to \underset{1}{G^{tor}} \to 1 \Big)$$

(iii) Suppose that G^{tor} is quasi-trivial. Then for K'/K finite extension, we have $\mathrm{cor}_{K'/K}\,\mathrm{res}_{K'/K}[C] = [K':K][C]$

where $[C] \in H^1(K, M^{tor} \to G^{tor})$.

Sketch of Proof:

① (i) $\Rightarrow$ we can assume G with G^{ss} simply connected and G^{tor} quasi-trivial.

② X has a zero cycle of degree 1.
$\Rightarrow$ X has a K_v-point for $v \in V_\infty$.

Borovoi's Thm

$\Rightarrow$ $\eta(X) = 0 \Leftrightarrow X(K) \neq \emptyset$

③ ① + (iii) $\eta(X) = 0$

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